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The Complements of Projective Plane Curves

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In this note, we shall study rational plane curves. Our subject is the logarithmic Kodaira dimension of a rational plane curve, introduced by Iitaka[1].

First of all, we recall the logarithmic Kodaira dimension. Let X be a nonsingular surface defined over the complex number field $\mathbf{C}$. We find a smooth completion $\overline{X}$ of X such that $D := \overline{X} - X$ is a divisor with simple normal crossings. Let K be a canonical divisor of $\overline{X}$. Then $\overline{P}_m(X) := \dim H^0(\overline{X}, m(K + D))$ is called the logarithmic m-genus of X. Then the logarithmic Kodaira dimension $\overline{\kappa}(X)$ of X is defined by $\kappa(K + D, \overline{X})$. It is easy to check that $\overline{P}_m(X)$ and $\overline{\kappa}(X)$ do not depend on the choice of $\overline{X}$ and D.

Let C be an irreducible curve in $\mathbf{P}^2$. We use the following notations:

$g(C)$: the genus of the normalization of C

$s(C)$: the number of singular points of C

$r(C)$: the number of cuspidal singular points of C

Assume $g(C) = 0$. Then there exists a canonical inclusion $\mathrm{Reg}(C) \longrightarrow \mathbf{P}^1$, where $\mathrm{Reg}(C)$ is the regular locus of C.

$t(C)$: the number of $\mathbf{P}^1 - \mathrm{Reg}(C)$.

We summarize known results by Wakabayashi [4].

(1) If $g(C) > 0$ and C is not a nonsingular elliptic curve, then $\bar{\kappa}(\mathbf{P}^2 - C) = 2$.

(2) If C is a nonsingular elliptic curve, then $\bar{\kappa}(\mathbf{P}^2 - C) = 0$.

(3) If $g(C) = 0$ and $s(C) \geqq 3$, then $\bar{\kappa}(\mathbf{P}^2 - C) = 2$.

(4) If $g(C) = 0$ and $s(C) = r(C) = 2$, then $\bar{\kappa}(\mathbf{P}^2 - C) \geqq 0$.

(5) If $g(C) = 0$, $s(C) = 1$ and $t(C) \geqq 3$, then $\bar{\kappa}(\mathbf{P}^2 - C) > 0$.

(6) If $g(C) = 0$, $s(C) = 1$ and $t(C) = 2$, then $\bar{\kappa}(\mathbf{P}^2 - C) \geqq 0$.

(7) If $g(C) = s(C) = 0$, then $\bar{\kappa}(\mathbf{P}^2 - C) = -\infty$.

Our results on the remaining cases are stated as follows.

Proposition 1. If $g(C) = 0$ and $r(C) = s(C) = 2$, then $\bar{\kappa}(\mathbf{P}^2 - C) > 0$.

Proposition 2. If $g(C) = 0$ and $s(C) = r(C) = 1$, then $\bar{\kappa}(\mathbf{P}^2 - C) \neq 0$.

Let a be an integer greater than 2, let $\delta, \gamma_1, \ldots, \gamma_a$ be complex numbers and let ε, γ_0 be nonzero complex numbers. Then the P_{ki}'s ($0 \leqq i \leqq a-1$, $0 \leqq k \leqq a$) are defined by the equations as follows:

$$\binom{a}{i}(\delta u + \varepsilon)^{a-i}(\gamma_0 + \gamma_1 u + \ldots + \gamma_a u^a)$$

$$= P_{ia} + P_{ia-1}u + \ldots + P_{i0}u^a + (\text{higher terms}),$$

where u is an indeterminate. Let (x,y,z) be a system of homogeneous coordinates in $\mathbf{P}^2$. Let $C_{a,\gamma_0,\ldots,\gamma_a,\delta,\varepsilon}$ be the curve defined by the equation

$$(y^{a-1}z - (\gamma_0 x^a + \gamma_1 x^{a-1}y + \ldots + \gamma_a y^a))^a z$$

$$+ \sum_{i=0}^{a-1} \sum_{k=0}^{a} P_{ik} x^k y^{a^2-ai+1-k} (y^{a-1}z - (\gamma_0 x^a + \gamma_1 x^{a-1}y + \ldots + \gamma_a y^a))^i$$

$$= 0.$$

Proposition 3. The curve $C_{a,\gamma_0,\ldots,\gamma_a,\delta,\varepsilon}$ has the following properties:

(1) $C_{a,\gamma_0,\ldots,\gamma_a,\delta,\varepsilon} - \{p\} \cong \mathbf{A}^1$, for some point p of C,

(2) $\overline{\kappa}(\mathbf{P}^2 - C_{a,\gamma_0,\ldots,\gamma_a,\delta,\varepsilon}) = 1$.

Theorem 4. Let C be a projective plane curve satisfying the conditions:

(1) $C - \{p\} \cong \mathbf{A}^1$, for some point p of C,

(2) $\overline{\kappa}(\mathbf{P}^2 - C) = 1$.

Then C is isomorphic to $C_{a,\gamma_0,\ldots,\gamma_a,\delta,\varepsilon}$, up to projective equivalence for some $(a,\gamma_0,\ldots,\gamma_a,\delta,\varepsilon)$.

Let C be a projective plane curve such that $C - \{p\} \cong \mathbf{A}^1$, for some point p of C. This curve was studied by Yoshihara[5]. His results are as follows:

(1) If $\deg C \geqq 3\ \mathrm{mult}_p(C)$, then $\bar{\kappa}(\mathbf{P}^2 - C) = 2$,

(2) there exist no curves such that $\deg C = 6$ and $\mathrm{mult}_p(C) = 2$.

Now, we have the following:

Proposition 5. Let C be as above. Then,

$$\deg C \leqq 3\ \mathrm{mult}_p(C) + 2.$$

Furthermore, if $\deg C > 192$, then

$$\deg C < 3\ \mathrm{mult}_p(C).$$

By the above result, we naturally have the following

Conjecture: Under the above notations,

$$\deg C < 3\ \mathrm{mult}_p(C).$$

Finally, we explain the outline of the proofs of Theorem 4 and the first part of Proposition 5.

The proof of Theorem 4: Let $\mu : \overline{X} \longrightarrow \mathbf{P}^2$ be a composite of blowing-ups such that $D:=\mu^{-1}(C)$ has only simple normal crossings. Assume that μ is the shortest among such birational morphisms, We set $X:= \mathbf{P}^2 - C = \overline{X} - D$. We denote by $\Pi: X \longrightarrow \Delta$ a rational map associated with $|n(K(\overline{X}) + D)|$ for sufficiently large n. Since $\overline{\kappa}(\mathbf{P}^2 - C) = 1$, we can apply Kawamata's results[2]. Then, Π is a morphism and a general fiber of $\Pi_{|X}$ is $\mathbf{G}_m$ or an elliptic curve. Since X is affine, a general fiber of $\Pi_{|X}$ is $\mathbf{G}_m$, whence a general fiber of Π is $\mathbf{P}^1$. Hence, there exist a Hirzebruch surface $\overline{Y}$ and a birational morphism $\rho: \overline{X} \longrightarrow \overline{Y}$ such that $\Pi\cdot\rho^{-1}$ is a morphism. We put $\psi = \Pi\cdot\rho^{-1}$ and denote by ℓ a general fiber of ψ.

By taking a suitable $\overline{Y}$, we may assume that $\Gamma = \rho_*(D)$ is either

(i) a sum of a 2 - section and at most three fibers, or

(ii) a sum of two sections for the fibration ψ and at most three fibers.

Note that

(1) each irreducible component of D has a negative self-intersection number and

(2) the exceptional curve in D is unique.

It follows from (1) and (2) that Γ is a sum of two sections and three fibers. Using these facts, we conclude that $(X,\overline{X},D)$ is a resolution of $C_{a,\gamma_0,\ldots,\gamma_a,\delta,\varepsilon}$ for some $(a,\gamma_0,\ldots,\gamma_a,\delta,\varepsilon)$.

Q.E.D.

The proof of Proposition 5: We shall only prove the first part of Proposition 5. By a rather easy argument, we can obtain that $\deg C \leqq 3\, \mathrm{mult}_p(C) + 2$. Put $n := \deg C$ and $e := \mathrm{mult}_p(C)$. First, consider the shortest succession of blowing-ups

$$\bar{X}_0 \xleftarrow{f_1} \bar{X}_1 \longleftarrow \cdots \xleftarrow{f_s} \bar{X}_s := \bar{X}$$

such that (1) $X_0 = \mathbf{P}^2$, (2) the center p_i of the blowing-up f_i lie over $p = \mathrm{Sing}\,(C)$ and (3) $D = f^{-1}(C)$ has simple normal crossings, where $f = f_1 \ldots f_s$. We denote by e_i the multiplicity of C at the center of f_i. Note that $e_1 = e$. By the Plücker formula, we have

$$(n-1)(n-2) = \sum_{i=1}^{r} e_i(e_i - 1). \qquad (1)$$

By Yoshihara's result, we have $\bar{\kappa}(\mathbf{P}^2 - C) = 2$. Hence, we can use the following fact: Under the above situations,

$$4\, e(\mathbf{P}^2 - C) \geqq (K(\bar{X}) + D)^2,$$

where $e(\mathbf{P}^2 - C)$ is the Euler number of $\mathbf{P}^2 - C$ (cf. Sakai[3]).

In the present case, since C is a rational plane curve which has only one cuspidal singular point, we have $e(\mathbf{P}^2 - C) = 1$. Furthermore,

$$\begin{aligned}(K(\bar{X}) + D)^2 &= (K(\bar{X}), K(\bar{X}) + D) + (D, K(\bar{X}) + D)\\ &= (K(\bar{X}), K(\bar{X}) + D) - 2,\end{aligned}$$

where $(D, K(\bar{X}) + D) = -2$, because D is connected and the dual graph of D is a tree. Hence, we have

$$6 \geqq (K(\overline{X}) + D, K(\overline{X})). \tag{2}$$

We shall next compute $(K(\overline{X}), K(\overline{X}) + C')$, where C' is the proper transform of C. Since $(C'^2) = n^2 - \Sigma e_i^{\,2}$, we have $(K(\overline{X}), C') = -2 - n^2 + \Sigma e_i^{\,2}$. By (1), we have

$$(K(\overline{X}), K(\overline{X}) + C') = 9 - s + \Sigma e_i - 3n. \tag{3}$$

Note that

$$(K(\overline{X}), D - C') \geqq 0. \tag{4}$$

In fact, since f is a resolution of a cuspidal singular point, we see that one of the two irreducible components (except C') meeting the exceptional curve of the first kind in D has self-intersection number $\leqq -3$. From (1),(2),(3) and (4), we conclude that $n \leqq 3e + 2$. Q.E.D.

References

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